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Efficiency and optimized dimensions of annular fins of different cross-section shapes

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Abstract—Increasing the heat dissipation of annular fins at a defined magnitude of mass is the problem considered in this paper. Four different cross-section shapes are examined—constant thickness, constant area for heat flow, triangular and parabolic fin shapes. The fin efficiency together with the optimized dimensions are presented, which enables the design of the best fin for any practical use.

INTRODUCTION

ANNULAR fins are of great practical importance in compact heat exchangers, finned tubes, etc. For a given fin weight, the fin can dissipate various quantities of heat, depending on its shape and geometry. Optimizing the fin, namely finding the shape that would dissipate the maximum heat for a given weight, is an important requirement in fin design.

Gardner [1] used a set of idealizing assumptions and found the efficiency of various straight fins and spines. He also showed the efficiency of annular fins that have constant thickness and constant metal area for heat flow—rectangular and hyperbolic fins. His figures appeared in Eckert and Drake [2] and in some other books, but the last fin was incorrectly titled as a triangular cross-section fin.

For straight based fins the optimization problem was solved by Schmidt [3] and was confirmed by Duffin and McLain [4, 5]. They assumed that the minimum weight fin has a linear temperature distribution along its length. They also included the length-of-arc assumption that was dropped in Maday's approach [6].

Guceri and Maday [7] and Mikk [8] found the optimum fin thickness variation along the fin. This type of fin shape is complex and has little use in practice, mostly because of manufacturing problems. An annular fin should be treated in the same way as a straight fin—the optimum dimensions should be determined for a given fin shape. Brown [9] made use of Bessel functions to calculate the optimum dimensions of the most common annular fin shape—constant thickness.

In this paper, Brown's approach [9] has been adopted, the temperature profile, the efficiency and the optimum dimensions of four different shapes of annular fins (rectangular, triangular, hyperbolic and parabolic) are determined by solving numerically the differential equations.

THE TEMPERATURE PROFILE

Consider an annular fin the heat of which is supplied at the base with constant temperature T_0, and thickness $\delta(r)$. The analysis is based upon the following assumptions which are common to most of the previous investigations.

(1) The temperature profile is steady.

(2) The fin material is homogeneous—the thermal conductivity, k, and the density, ρ, are constant (their dependence on the temperature is negligible).

(3) The heat transfer coefficient, h, between the fin surface and the environment is constant.

(4) The temperatures of the fin base and the environment are constant.

(5) The fin thickness is small compared to its length, and the thermal conductivity is much larger than the heat transfer coefficient (Bi is very small). Therefore, the temperature gradients are only in the radial direction and the temperature gradients at the thickness may be neglected.

(6) The heat transferred through the edge of the fin is neglected compared to the heat removed from the entire surface of the fin.

In the cases in which the heat transferred from the fin surface to the environment by radiation and by free convection cannot be neglected, the total heat transfer coefficient depends upon the temperature difference and varies, therefore, along the fin. On those occasions, the validity of assumption 3 is questionable, but using the average heat transfer coefficient as constant around the entire fin gives reasonable accuracy for many practical purposes.

The heat balance for a control volume of length, dr, as shown in Fig. 1, is given by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(k\pi r\delta\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = 2h\pi r\theta\left[\frac{1}{4}\left(\frac{\mathrm{d}\delta}{\mathrm{d}r}\right)^2+1\right]^{1/2}. \quad (1)$$

NOMENCLATURE

- Bi Biot number, equation (13), $2hR_0/k$
- G volume integration, equation (11)
- h heat transfer coefficient [W m^{-2} °C^{-1}]
- k thermal conductivity [W m^{-1} °C^{-1}]
- L fin length [m]
- m_f fin parameter, $L\sqrt{(2h/k\delta_0)}$
- M fin mass [kg]
- $\bar{M}$ dimensionless mass, $M/\rho R_0^3$
- n constant for fin shape definition
- q_f heat dissipation, equation (7) [W]
- q_{f0} maximum heat dissipation, equation (8) [W]
- $\bar{q}$ dimensionless heat dissipation, $q_f/2h\pi R_0^2\theta_0$
- r radius [m]
- R_f constant for fin shape definition
- $\bar{R}$ dimensionless outer edge radius, R_1/R_0
- V volume of fin material [m^3]
- x dimensionless radius, r/R_0.

Greek symbols

- δ fin thickness [m]
- $\bar{\delta}$ dimensionless fin thickness, δ/δ_0
- η fin efficiency
- θ temperature excess of fin over fluid [°C]
- ρ material density [kg m^{-3}]
- ϕ dimensionless temperature, θ/θ_0.

Subscripts

- 0 fin base
- 1 outer edge of fin.

Introducing the normalized variables: $\bar{\delta} = \delta/\delta_0$, $\phi = \theta/\theta_0$, $x = r/R_0$, $\bar{R} = R_1/R_0$, and the fin parameter [1], $m_f = L\sqrt{(2h/k\delta_0)}$, into equation (1) results in

$$\frac{d^2\phi}{dx^2} + \left[\frac{1}{x} + \frac{1}{\bar{\delta}}\frac{d\bar{\delta}}{dx}\right]\frac{d\phi}{dx} - \frac{m_f^2}{\bar{\delta}(\bar{R}-1)^2}\left[\frac{1}{4}\left(\frac{d\bar{\delta}}{dx}\right)^2\left(\frac{\delta_0}{R_0}\right)^2 + 1\right]^{1/2}\phi = 0. \quad (2)$$

The fins differ from each other by their cross-section profile—variation of the fin thickness along the fin. Four common fin shapes can be defined by a single equation

$$\bar{\delta} = \left(\frac{R_f - x}{R_f - 1}\right)^n. \quad (3)$$

$n = 0$, $R_f = \bar{R}$, represents the constant thickness fin which has a rectangular shape. $n = -1$, $R_f = 0$, corresponds to the fin with a constant area for heat flow—hyperbolic shape. The hyperbolic fin has a sharp edge at infinity but in practice, it is cut off at a distance R_1 from the axis of symmetry. $n = 1$, $R_f = \bar{R}$, describes the triangular fin, starting with a thickness of δ_0 at the base (R_0) and tip ($\delta = 0$) at the end (R_1). The surfaces of the triangular fin are straight at an angle that depends on the base thickness, δ_0, and length, L. $n = 2$, $R_f = \bar{R}$, creates the parabolic fin that, as the triangular fin, has a base thickness of δ_0, and sharp edge at R_1.

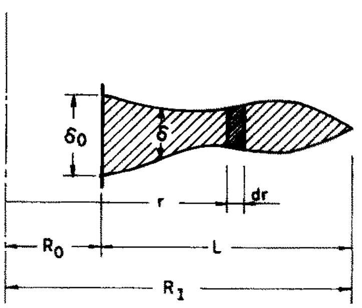

FIG. 1. Control volume for heat balance.

Introducing the normalized fin thickness, equation (3), and its derivative into equation (2) yields

$$\frac{d^2\phi}{dx^2} + \left(\frac{1}{x} - \frac{n}{R_f - x}\right)\frac{d\phi}{dx} - \frac{m_f^2}{(\bar{R}-1)^2}\left[\frac{1}{4}\left(\frac{n}{R_f - x}\right)^2\left(\frac{\delta_0}{R_0}\right)^2 + \left(\frac{R_f - 1}{R_f - x}\right)^{2n}\right]^{1/2}\phi = 0 \quad (4)$$

which can be solved with boundary conditions

$$\phi = 1 \quad \text{at } x = 1$$
$$\frac{d\phi}{dx} = 0 \quad \text{at } x = \bar{R}. \quad (5)$$

The first boundary condition is obvious in regard to assumption 4, and the second is derived from assumption 6.

The dimensionless temperature, ϕ, is a function of three kinds of normalized variables, n and R_f, due to the chosen fin shape, m_f, due to the heat transfer quality and δ_0/R_0, $\bar{R}$, x due to the fin geometry. Considering x as the only independent variable and keeping the other variables constant, the second-order differential equation (4) can be solved by various simple numerical methods for the four fin shapes. Using Bessel's functions, the equation can be solved analytically for rectangular and hyperbolic fins [1].

FIN EFFICIENCY

The fin efficiency is defined as the ratio between the heat removed by the fin and the heat that would have been removed if the entire surface area of the fin had been maintained at the base temperature

$$\eta = q_f/q_{f0}. \quad (6)$$

Knowing the temperature profile by solving equation (4) and using Fourier's law at the fin base, the amount of heat which enters the fin and that which is removed to the environment can be calculated

$$q_f = -2\pi k\delta_0\theta_0 \left.\frac{d\phi}{dx}\right|_{x=1}. \quad (7)$$

If the surface temperature of the fin is constant at the base temperature, the problem of calculating the heat that dissipates from the fin is reduced to the calculation of the surface area, by using Newton's law

$$q_{f0} = 4\pi h R_0^2\theta_0 \times \int_1^{\bar{R}} \left[\frac{1}{4}n^2\frac{(R_f-x)^{2n-2}}{(R_f-1)^{2n}}\left(\frac{\delta_0}{R_0}\right)^2+1\right]^{1/2} x\,dx. \quad (8)$$

Substituting equations (7) and (8) into efficiency definition (6) yields

$$\eta = \frac{-\left.\dfrac{d\phi}{dx}\right|_{x=1}}{\dfrac{m_f^2}{(\bar{R}-1)^2}\displaystyle\int_1^{\bar{R}}\left[\frac{1}{4}n^2\frac{(R_f-x)^{2n-2}}{(R_f-1)^{2n}}\left(\frac{\delta_0}{R_0}\right)^2+1\right]^{1/2} x\,dx}. \quad (9)$$

It follows from the above equation (9) that the temperature distribution or, at least, the first derivative of the temperature at the fin base, enables the fin efficiency to be calculated. The fin efficiency depends on the same normalized variables as the temperature profile. Notice, however, that for the rectangular fin, the efficiency and the temperature profile do not depend on δ_0/R_0.

The equations for the temperature gradient and fin efficiency derived in this paper are quite general for many practical annular fins. The sharp ended fins—triangular and parabolic—can be cut at any length by given different values to R_f, $R_f \leqslant \bar{R}$. The slope of the fin surface is determined by the value of δ_0 and R_f, but the real end of the fin is determined by the value of $\bar{R}$.

The numerical solutions of the fin efficiency for the four fin shapes are shown in Figs. 2–5 vs the fin parameter, m_f. Figures 2 and 3 are identical to the analytical solution of Gardner [1]. The fin efficiency is obviously a maximum when the fin length is zero. As the fin parameter increases, the fin efficiency decreases, first sharply and later only slightly. The lines representing $\bar{R} = 1$ in Figs. 2 and 3 relate to the straight base fins as $R_0 \to \infty$. At the same parameter values, the rectangular shape fin has the highest efficiency and the parabolic shape fin the lowest efficiency, due to enlargement of the convection area and reduction of the conduction area. The hyperbolic fin is exceptional because the value of $\bar{R}$ does not have any influence on the surface slope. For $\bar{R} = 1$, the efficiency is almost the same as for the rectangular fin, but as $\bar{R}$ increases, the fin becomes more and more like the sharp ended fin—the decrease of the efficiency is emphasized. The group δ_0/R_0 influences the fin efficiency (except for the rectangular fin) only slightly at low $\bar{R}$ for growth from 0.01 to 0.6 by a factor of 60.

OPTIMIZED FINS

The designer should be interested in and choose the best shape of fin in regard to economical and manufacturing considerations but, after the shape has

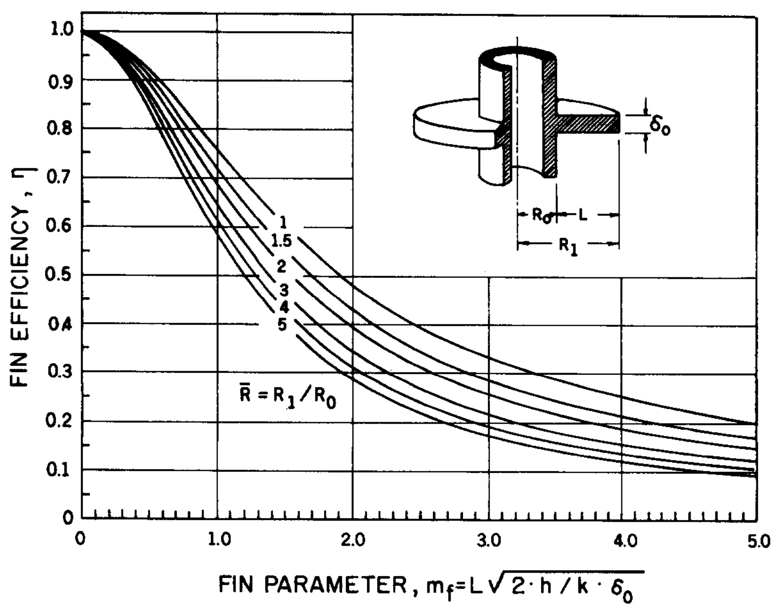

FIG. 2. The efficiency of rectangular shape fin, $n = 0$.

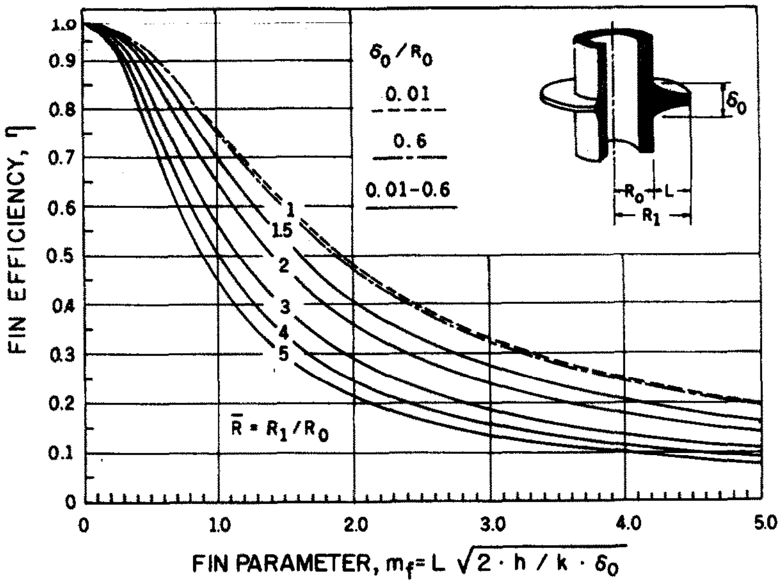

FIG. 3. The efficiency of hyperbolic shape fin, $n = -1$.

been chosen, the designer should aspire to the optimized dimensions.

The optimized dimensions of the fin can be found in either one of two ways: the maximum amount of heat dissipation for a given quantity of weight or the minimum weight for dissipating a given quantity of heat. In this paper the first way has been used.

The heat removed by the fin was found in equation (7). Concerning assumption 2, the mass of the fin is obtained by the fin volume

$$M = \rho V = 2\rho\pi R_0^2\delta_0 \int_1^{\bar{R}} \bar{\delta} x \, dx = 2\rho\pi R_0^2\delta_0 G \quad (10)$$

where G is the result of the integration and has a general expression for the four fin shapes

$$G = \frac{(\bar{R}-1)(\bar{R}+n+1)}{(n+1)(n+2)}. \quad (11)$$

The heat dissipation is normalized by $2h\pi R_0^2\,\theta_0$ and the fin mass is normalized by ρR_0^3. The dimensionless heat dissipation per mass is written as

$$\frac{\bar{q}}{\bar{M}} = -\frac{2}{G\pi}\frac{1}{Bi}\left.\frac{d\phi}{dx}\right|_{x=1} \quad (12)$$

where

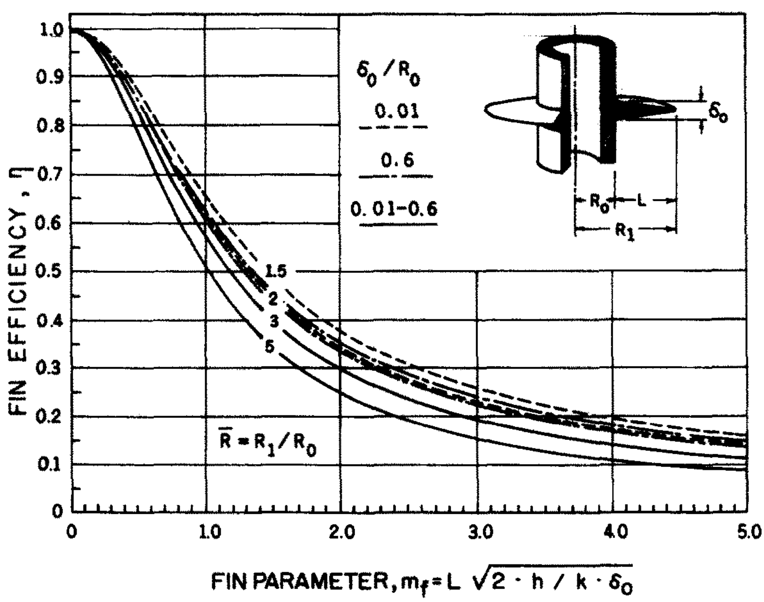

FIG. 4. The efficiency of triangular shape fin, $n = 1$.

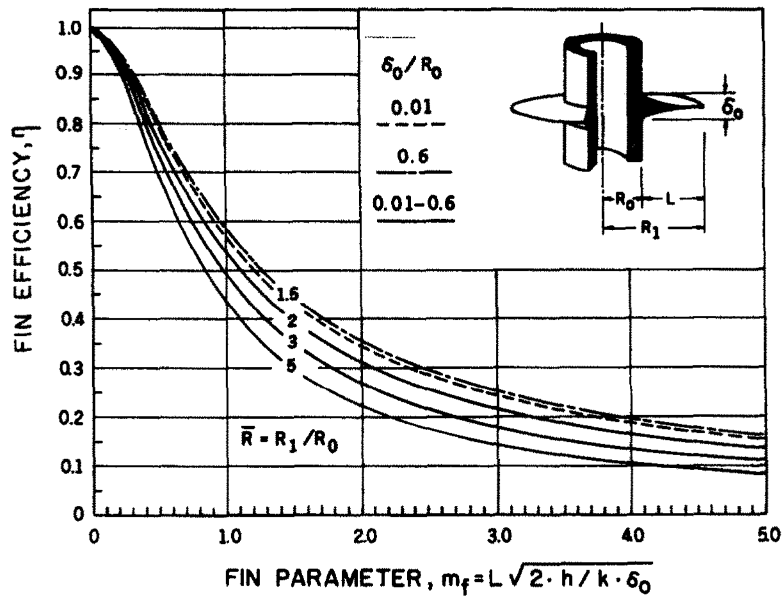

FIG. 5. The efficiency of parabolic shape fin, $n = 2$.

$$Bi = \frac{2hR_0}{k}. \tag{13}$$

Considering $\bar{R}$ as the only independent variable, the dimensionless heat per mass can be found numerically by solving equation (4). Equation (12) is plotted in Fig. 6 vs $\bar{R}$ for $Bi = 0.01$, $\bar{M} = 0.1$ and 0.5, and for the four fin shapes. Notice that from the definition of the normalized mass, $\bar{M}$, and $\bar{R}$, the group δ_0/R_0 is dictated.

Obviously, from Fig. 6, there is a maximum value of the dimensionless heat dissipation per mass, $\bar{q}/\bar{M}$, for any fin shape and given Bi, mass and $\bar{R}$ (consequently δ_0/R_0). Although the heat dissipated increases by increasing $\bar{R}$, $\bar{q}/\bar{M}$ decreases after the maximum occurs. This is due to the fact that the increase of mass is greater than the heat dissipation. The dimensions of the fin, where this maximum occurs, are the optimized dimensions. The parabolic fin dissipates more heat than the others at the same mass by being a longer fin. A heavier fin dissipates more heat at larger values of optimized $\bar{R}$.

In the same way as Fig. 6 was plotted, the lines that describe the optimum dimensions could be found by accumulation of the maximum values for slight and continuous increases of mass. $(\bar{q}/\bar{M})Bi$ and $(\delta_0/R_0)/Bi$ are plotted vs $\bar{R}$ in Fig. 7. The dimensionless heat dissipation per mass multiplied by the Biot number

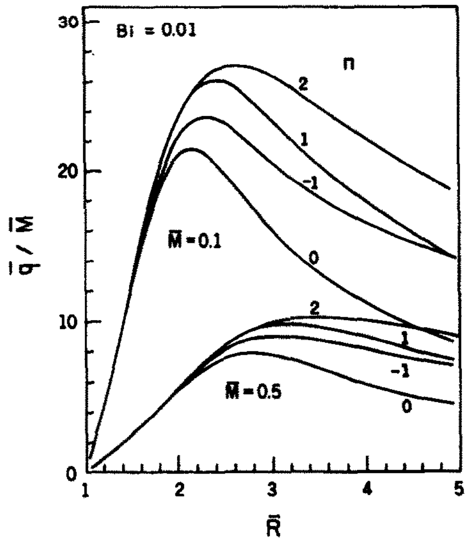

FIG. 6. The dimensionless heat dissipation per mass for constant Biot number and mass.

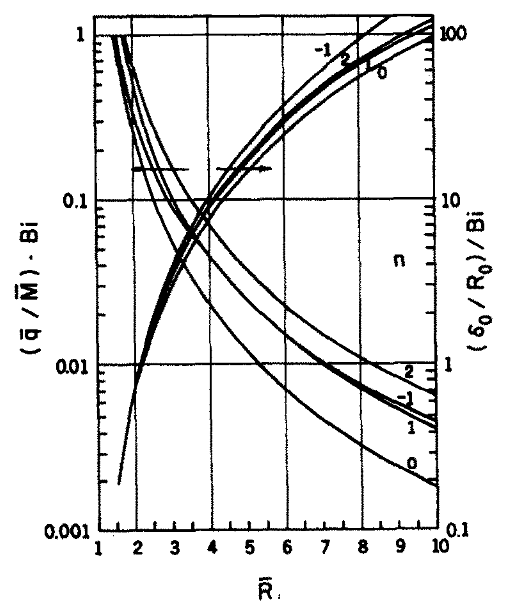

FIG. 7. The optimized dimensions of rectangular, hyperbolic, triangular and parabolic shape fins.

appeared to be unique and is a function of only the dimensionless outer edge radius, the same as δ_0/R_0 divided by the Biot number, for the optimized fin. Figure 7 is the concluding figure that enables the designer to use the optimized dimensions from any point of view.

CONCLUSION

As a result of the foregoing analysis, the second-order differential equation, derived from the heat balance, has been solved numerically to illustrate the efficiency and optimized fin dimensions. It has been shown clearly that fins with a sharp edge and a sharper reduced thickness have lower efficiencies and higher quantities of heat dissipation per mass. The parabolic fin has the best performance among the four examined fin shapes, although it has more manufacturing problems. The designer should choose the best fin in relation to economical considerations.

To sum up Fig. 7 should be an important tool for the designer. The use of the figure is emphasized by the next example. Circular fins should be combined on a given pipe with known radius, R_0, and in a known environment, h. After the fin material has been chosen, the Biot number can be calculated. If the maximum fin length is determined by geometric considerations, Fig. 7 gives $(\delta_0/R_0)/Bi$ and $(\bar{q}/\bar{M})Bi$ for any fin shape. The base fin thickness has been found and by using equations (10) and (11) the normalized mass is calculated. Because the heat dissipation is normalized by known variables, the heat dissipated from the optimized fin can be calculated in comparison to the desirable heat dissipation.

The optimization that has been made in this paper relates to a single fin. The distance between fins significantly influences the heat transfer coefficient. For real industrial systems, more attention must be paid to optimization of fin arrays that are made from individual optimized fins.

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EFFICACITE ET DIMENSIONS OPTIMISEES DES AILETTES ANNULAIRES DONT LES SECTIONS DROITES SONT DE FORME DIFFERENTE

Résumé—On considère le problème d'accroissement du transfert de chaleur par des ailettes annulaires pour une masse donnée de matière. On étudie quatre sections droites différentes : épaisseur constante, section constante de passage de la chaleur, forme triangulaire et parabolique. On présente l'efficacité et les dimensions optimisées de l'ailette, ce qui facilite la conception des meilleures ailettes dans un cas pratique.

WIRKUNGSGRAD UND OPTIMIERTE ABMESSUNGEN VON RINGFÖRMIGEN RIPPEN UNTERSCHIEDLICHER QUERSCHNITTSFORM

Zusammenfassung—In dieser Arbeit wird die Erhöhung der Wärmeabgabe von ringförmigen Rippen bei konstanter Rippenmasse untersucht. Dabei wurden 4 verschiedene Querschnittsformen betrachtet : Rippen mit konstanter Dicke, mit konstanter wärmeübertragender Oberfläche, mit dreieckigem oder parabolischem Querschnitt. Der Rippenwirkungsgrad wird zusammen mit den optimierten Abmessungen dargestellt, was die Konstruktion der günstigsten Rippenform für den jeweiligen Anwendungsfall ermöglicht.

ЭФФЕКТИВНОСТЬ И ОПТИМАЛЬНЫЕ РАЗМЕРЫ КОЛЬЦЕВЫХ РЕБЕР С ПОПЕРЕЧНЫМИ СЕЧЕНИЯМИ РАЗЛИЧНОЙ ФОРМЫ

Аннотация—Исследуется задача увеличения рассеяния тепла от кольцевых ребер при заданной их массе. Рассматриваются четыре различных случая: постоянная толщина, постоянная площадь теплового потока, треугольная и параболическая формы ребер. Определены эффективность и оптимальные размеры, позволяющие конструировать ребра, наиболее подходящие для практического использования.